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Some indefinite cases of spectral problems for canonical systems of difference equations

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Abstract

An approach by operator identities is used to investigate some direct and inverse problems of spectral theory for canonical systems of difference equations in the indefinite case. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The operator identity $AS - SA^* = \mathrm{i}(\Phi_1\Phi_2^* + \Phi_2\Phi_1^*)$ provides the underlying structure for both interpolation theory and the spectral theory of canonical differential and difference systems [4,5]. A.L. Sakhnovich [3] has shown that indefinite cases of interpolation theory arise when the condition $S \geqslant 0$ is replaced by $S = S^*$. In [3], the Nevanlinna representation of a function having positive imaginary part in the upper half-plane is replaced by the integral representation of a generalized Nevanlinna function by Kreĭn and Langer [2]. In this paper, we explore problems of spectral theory in the same situation. We use spaces with indefinite metrics [1].

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For simplicity, we assume the most concrete case of systems of matrix difference equations.

Let m and n be positive integers, $n = 2m$. Consider a system of difference equations

$$\begin{cases} Y(k,z) - Y(k-1,z) = \mathrm{i}zJq(k)Y(k-1,z), \quad k = 1,\dots,N, \\ D_2Y_1(0,z) + D_1Y_2(0,z) = 0, \end{cases} \tag{1.1}$$

where

$$J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \quad Y(k,z) = \begin{bmatrix} Y_1(k,z) \\ Y_2(k,z) \end{bmatrix}, \quad k = 0,\dots,N,$$

and $q(1),\dots,q(N)$ are $n \times n$ matrices such that

$$q(k) = q(k)^*, \quad q(k)Jq(k) = 0, \qquad k = 1,\dots,N. \tag{1.2}$$

We assume that the values of $Y_1(k,z)$ and $Y_2(k,z)$ belong to $\mathbb{C}^m$, and that D_1 and D_2 are $m \times m$ matrices such that

$$D_1D_2^* + D_2D_1^* = 0 \quad \text{and} \quad D_1D_1^* + D_2D_2^* = I_m.$$

The monodromy matrix associated with (1.1) is the $n \times n$ matrix-valued function $W(k,z)$ satisfying

$$\begin{cases} W(k,z) - W(k-1,z) = \mathrm{i}zJq(k)W(k-1,z), \quad k = 1,\dots,N, \\ W(0,z) = I_n, \end{cases} \tag{1.3}$$

that is,

$$W(k,z) = [I_n + \mathrm{i}zJq(k)]\cdots[I_n + \mathrm{i}zJq(2)][I_n + \mathrm{i}zJq(1)], \quad k = 1,\dots,N. \tag{1.4}$$

As in the positive case (see [4, Lemma 8.1.1]), the identities

$$\sum_{k=0}^{N-1} W^*(k,\zeta)q(k+1)W(k,z) = \mathrm{i}\,\frac{J - W^*(N,\zeta)JW(N,z)}{z - \bar{\zeta}} \tag{1.5}$$

and

$$W(k,z)JW^*(k,\bar{z}) = J$$

are readily verified. Each $c \in \mathbb{C}^m$ determines a solution of (1.1) given by

$$Y(k,z) = W(k,z)\begin{bmatrix} D_1^* \\ D_2^* \end{bmatrix} c, \quad k = 0,\dots,N. \tag{1.6}$$

The kernel (1.5) has a finite number of negative squares determined by the signatures of the selfadjoint matrices $q(1),\dots,q(N)$. Many parts of the theory of operator-valued functions are valid in this generality [1]. Furthermore, in our case the functions involved have the concrete form (1.4), allowing explicit calculations that are not possible in general.

The direct and inverse problems of spectral theory for the boundary problem (1.1) are formulated in terms of a “Fourier” transform V_N, which is defined using the “eigenfunctions” of the system given by (1.6). The domain of V_N is the space $\ell^2(q, N)$ of vectors

$$F = \begin{bmatrix} F(0) \\ \vdots \\ F(N-1) \end{bmatrix}, \qquad F(0), \ldots, F(N-1) \in \mathbb{C}^n,$$

in the inner product

$$\langle F, G\rangle_{\ell^2(q,N)} = \sum_{k=0}^{N-1} G(k)^* q(k+1)F(k), \quad F, G \in \ell^2(q, N).$$

For any $F \in \ell^2(q, N)$,

$$V_N F = f(z)$$

is defined by

$$f(z) = \sum_{k=0}^{N-1} \begin{bmatrix} D_1 & D_2 \end{bmatrix} W^*(k, \bar{z})q(k+1)F(k). \tag{1.7}$$

Here $f(z)$ is a polynomial in z of degree at most $N-1$ with values in $\mathbb{C}^m$. In (1.7) we take u to be a complex variable. As will be seen, the bar over z in (1.7) is needed in the indefinite case. In the positive case ($q(k) \geqslant 0$, $k = 1, \ldots, N$), z can be taken to be a real variable, and the bar is absent in the counterpart to (1.7) in [4, (8.1.9)]; in this case, spectral data for (1.1) consists of a nondecreasing $m \times m$ matrix-valued function $\tau(u)$ of real u such that $V_N : \ell^2(q, N) \to L^2(\tau)$ is an isometry. In contrast, the definition of spectral data in the indefinite case uses nonreal points.

By *spectral data* for a boundary problem (1.1) we mean a tuple

$$\boldsymbol{\tau} = \{\tau, \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}, \tag{1.8}$$

where $\tau(u)$ is a function of real u of bounded variation whose values are selfadjoint $m \times m$ matrices such that $\int_{-\infty}^{\infty} |u|^p \|\mathrm{d}\tau(u)\| < \infty$, $p = 0, 1, \ldots, 2N-2$, and $\mathfrak{F}_1, \ldots, \mathfrak{F}_\nu$ are forms of the type described below, such that

$$V_N : \ell^2(q, N) \to L^2(\boldsymbol{\tau}, N) \tag{1.9}$$

is an isometry.

We explain the notation used in (1.8)–(1.9). For each $k = 1, \ldots, \nu$, we assume that λ_k is a complex number, $R_k(\lambda)$ is a polynomial whose values are $m \times m$ matrices with $R_k(0) = 0$, and

$$\begin{cases} \mathfrak{F}_k(f(z), g(z)) = \operatorname{Res}_{\lambda=\lambda_k}\left[g^*(\bar{\lambda})R_k\left(\frac{1}{\lambda-\lambda_k}\right) f(\lambda)\right], \\ \widehat{\mathfrak{F}}_k(f(z), g(z)) = \operatorname{Res}_{\lambda=\bar{\lambda}_k}\left[g^*(\bar{\lambda})R_k^*\left(\frac{1}{\bar{\lambda}-\lambda_k}\right) f(\lambda)\right], \end{cases} \tag{1.10}$$

for all polynomials $f(z)$ and $g(z)$ of degree at most $N-1$ with matrix values which are compatible for the required multiplications. By $L^2(\boldsymbol{\tau}, N)$ we understand the space of polynomials of degree at most $N-1$ with values in $\mathbb{C}^m$, in the inner product defined by

$$\langle f(z), g(z)\rangle_{L^2(\boldsymbol{\tau},N)} = \int_{-\infty}^{\infty} g^*(u)[\mathrm{d}\tau(u)]f(u) + \sum_{k=1}^{\nu} \left[\mathfrak{F}_k(f(z), g(z)) + \widehat{\mathfrak{F}}_k(f(z), g(z))\right] \tag{1.11}$$

for all elements of the space. The inner product is linear and symmetric but in general may be indefinite or degenerate. To verify symmetry of the inner product, write $R_k(\lambda) = \tau_{k1}\lambda + \tau_{k2}\lambda^2 + \cdots$, where $\tau_{k1}, \tau_{k2}, \ldots$ are $m \times m$ matrices and only a finite number are nonzero. Then the two expressions in (1.10) become

$$\mathfrak{F}_k(f(z), g(z)) = \left\{ g^*(\bar{\lambda})\tau_{k1}f(\lambda) + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[g^*(\bar{\lambda})\tau_{k2}f(\lambda)\right] + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[g^*(\bar{\lambda})\tau_{k3}f(\lambda)\right] + \cdots \right\}\Bigg|_{\lambda=\lambda_k},$$

$$\widehat{\mathfrak{F}}_k(f(z), g(z)) = \left\{ g^*(\bar{\lambda})\tau_{k1}^*f(\lambda) + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[g^*(\bar{\lambda})\tau_{k2}^*f(\lambda)\right] + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[g^*(\bar{\lambda})\tau_{k3}^*f(\lambda)\right] + \cdots \right\}\Bigg|_{\lambda=\bar{\lambda}_k}.$$

The proof of symmetry follows from these identities by algebraic calculations. We remark that it is possible to allow $\nu = \infty$, and in this case we must add a condition to assure convergence of the sum in (1.11).

In the simplest case of (1.8), $R_k(\lambda) = \tau_k\lambda$ for all k, and then (1.11) takes the form

$$\langle f(z), g(z)\rangle_{L^2(\boldsymbol{\tau},N)} = \int_{-\infty}^{\infty} g^*(u)[\mathrm{d}\tau(u)]f(u) + \sum_{k=1}^{\nu} \left[g^*(\bar{\lambda}_k)\tau_k f(\lambda_k) + g^*(\lambda_k)\tau_k^* f(\bar{\lambda}_k)\right]. \tag{1.12}$$

In this case, when $\nu = \infty$, the condition which is needed to insure convergence of the sums in (1.12) is $\sum_{k=1}^{\infty} |\lambda_k|^p \|\tau_k\| < \infty$, $p = 0, 1, \ldots, 2N-2$.

The boundary problem (1.1) has the same spectral data sets (1.8) as the problem

$$\begin{cases} Y(k,z) - Y(k-1,z) = \mathrm{i}zJ\tilde{q}(k)Y(k-1,z), \quad k = 1,\ldots,N, \\ Y_1(0,z) = 0, \end{cases} \tag{1.13}$$

where

$$\tilde{q}(k) = Uq(k)U^*, \quad k = 1, \ldots, N, \qquad U = \begin{bmatrix} D_2 & D_1 \\ D_1 & D_2 \end{bmatrix}.$$

The proof of this assertion is an obvious modification of the positive case [4, Theorem 8.1.1]. Accordingly, without loss of generality we may always choose $\begin{bmatrix} D_1 & D_2 \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix}$. Then (1.7) becomes

$$f(z) = \sum_{k=0}^{N-1} \begin{bmatrix} 0 & I_m \end{bmatrix} W^*(k, \bar{z}) q(k+1) F(k). \tag{1.14}$$

The direct problem of spectral theory is to find all spectral data sets (1.8) for a given boundary problem (1.1) (with $\begin{bmatrix} D_1 & D_2 \end{bmatrix} = [0 \quad I_m]$). The inverse problem is to recover $q(k)$, $k = 1, \ldots, N$, from given spectral data.

2. Operator identities and their applications to direct and inverse spectral problems

We recall how solutions of the operator identity

$$\begin{cases} AS - SA^* = \mathrm{i}(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*), \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}), \end{cases} \tag{2.1}$$

with

$$S = S^* \tag{2.2}$$

yield an environment for the spectral theory of boundary problems. The algebraic aspects of the constructions follow the positive case [4].

For the underlying spaces in (2.1) we choose $\mathfrak{H} = \mathbb{C}^m \oplus \cdots \oplus \mathbb{C}^m$ (N summands) and $\mathfrak{G} = \mathbb{C}^m$ in the Euclidean metrics. Let $S_0, \ldots, S_{2N-2}$ be selfadjoint $m \times m$ matrices. Set

$$\begin{cases} A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I_m & 0 & \cdots & 0 & 0 \\ 0 & I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 \end{bmatrix}_{N \times N}, \\ \\ S = \begin{bmatrix} S_0 & S_1 & \cdots & S_{N-1} \\ S_1 & S_2 & \cdots & S_N \\ \vdots & \vdots & \ddots & \vdots \\ S_{N-1} & S_N & \cdots & S_{2N-2} \end{bmatrix}_{N \times N}, \end{cases} \tag{2.3}$$

and

$$\Phi_1 = -\mathrm{i}\begin{bmatrix} 0 \\ S_0 \\ \vdots \\ S_{N-2} \end{bmatrix}_{N\times 1}, \qquad \Phi_2 = \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{N\times 1}. \tag{2.4}$$

Then the identities (2.1) and (2.2) are satisfied.

For any $N \times N$ block matrix $M = [M_{ij}]_{i,j=1}^{N}$, let $M^{(k)} = [M_{ij}]_{i,j=1}^{k}$ be the upper left $k \times k$ block of M ($k = 1, \ldots, N$). We write E_k for the natural embedding of $\mathfrak{H}^{(k)} = \mathbb{C}^m \oplus \cdots \oplus \mathbb{C}^m$ (k summands) into $\mathfrak{H}^{(N)} = \mathfrak{H}$. Thus $M^{(k)} = E_k M E_k^*$. For later use we note that if M is upper triangular, then

$$E_k M^{(k)} = M E_k, \quad k = 1, \ldots, N. \tag{2.5}$$

The proof is a simple calculation, which we omit.

Theorem 2.1. *Let a solution of* (2.1), (2.2) *be given in the form* (2.3), (2.4). *Assume that $S^{(1)}, \ldots, S^{(N)}$ are invertible. Consider the boundary problem*

$$\begin{cases} Y(k,z) - Y(k-1,z) = \mathrm{i}zJq(k)Y(k-1,z), & k = 1, \ldots, N, \\ Y_1(0,z) = 0, \end{cases} \tag{2.6}$$

in which the selfadjoint $n \times n$ matrices $q(1), \ldots, q(N)$ are defined by

$$q(k) = \sigma(k) - \sigma(k-1), \quad k = 1, \ldots, N, \tag{2.7}$$

where

$$\begin{cases} \sigma(0) = 0, \qquad \sigma(k) = \Pi^* E_k \big(S^{(k)}\big)^{-1} E_k^* \Pi, \quad k = 1, \ldots, N, \\ \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}. \end{cases} \tag{2.8}$$

Then $q(1), \ldots, q(N)$ satisfy (1.2), *and the monodromy matrix for the system* (2.6) *is given by*

$$W(k,z) = I_{2m} + \mathrm{i}zJ\Pi^* E_k \big(S^{(k)}\big)^{-1} \big(I_{km} - zA^{(k)}\big)^{-1} E_k^* \Pi, \quad k = 1, \ldots, N.$$

Proof. As in the corresponding result for the positive case [4, Theorem 8.1.2], Theorem 2.1 follows from the factorization theorem for S-nodes [5, Theorem 2.1, p. 21]. □

In Theorem 2.1, the hypothesis that the operator matrices $S^{(1)}, \ldots, S^{(N)}$ are invertible can be formulated in another way. In fact, for a selfadjoint block matrix $M = [M_{ij}]_{i,j=1}^{N}$, a necessary and sufficient condition that $M^{(1)}, \ldots, M^{(N)}$ are invertible is that

$$M = LDL^*,$$

where L and D are invertible $N \times N$ block matrices, L is block lower triangular, and D is block diagonal. The sufficiency part of this assertion is straightforward. Necessity is proved by induction on N using a standard factorization of a selfadjoint block operator matrix, namely,

$$\begin{bmatrix} T & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^*T^{-1} & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & C - B^*T^{-1}B \end{bmatrix} \begin{bmatrix} I & T^{-1}B \\ 0 & I \end{bmatrix} \tag{2.9}$$

whenever T^{-1} exists.

The main results of this section, Theorems 2.2 and 2.3, generalize results in [4, Chapter 8] for the positive case. The first reduces the inverse problem of spectral theory to an interpolation problem.

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Suppose further that S has the form*

$$\begin{aligned} S = & \int_{-\infty}^{\infty} (I - Au)^{-1} \Phi_2 \, [\mathrm{d}\tau(u)] \, \Phi_2^*(I - A^*u)^{-1} \\ & + \sum_{k=1}^{\nu} \Big[\mathfrak{F}_k \left(\Phi_2^*(I - A^*z)^{-1}, \Phi_2^*(I - A^*z)^{-1} \right) \\ & \qquad + \widehat{\mathfrak{F}}_k \left(\Phi_2^*(I - A^*z)^{-1}, \Phi_2^*(I - A^*z)^{-1} \right) \Big], \end{aligned} \tag{2.10}$$

where $\boldsymbol{\tau} = \{\tau, \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}$ *is a tuple as in Section* 1 *such that a space* $L^2(\boldsymbol{\tau}, N)$ *is defined. Then the boundary problem* (2.6) *has spectral data* $\boldsymbol{\tau}$*.*

To see the nature of the interpolation problem (2.10), consider the simplest non-trivial case of data $\boldsymbol{\tau} = \{\tau, \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}$ where the discrete part (1.10) is determined by linear polynomials $R_k(\lambda) = \tau_k \lambda$, $k = 1, \ldots, \nu$. By (2.3) and (2.4),

$$\Phi_2^*(I - A^*z)^{-1} = \begin{bmatrix} I_m & I_m z & I_m z^2 & \cdots & I_m z^{N-1} \end{bmatrix}.$$

Since $S = [S_{i+j}]_{i,j=0}^{N-1}$, a short calculation shows that in this special case (2.10) takes the form

$$S_\ell = \int_{-\infty}^{\infty} u^\ell \, \mathrm{d}\tau(u) + \sum_{k=1}^{\nu} \left(\tau_k \lambda_k^\ell + \tau_k^* \bar{\lambda}_k^\ell \right), \qquad \ell = 0, \ldots, 2N-2.$$

The general case of the interpolation problem for polynomials $R_k(\lambda) = \tau_{k1}\lambda + \tau_{k2}\lambda^2 + \cdots$, $k = 1, \ldots, \nu$, has additional terms in the discrete part and is otherwise the same.

Proof of Theorem 2.2. Let

$$F = \begin{bmatrix} F(0) \\ \vdots \\ F(N-1) \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} G(0) \\ \vdots \\ G(N-1) \end{bmatrix} \tag{2.11}$$

be arbitrary elements of $\ell^2(q, N)$, and let $f(z)$ and $g(z)$ be their transforms as defined by (1.14) and its counterpart with F replaced by G. We must show that

$$\langle f(z), g(z) \rangle_{L^2(\boldsymbol{\tau}, N)} = \langle F, G \rangle_{\ell^2(q,N)}. \tag{2.12}$$

By linearity, it is sufficient to treat the case where

$$F(j)=\begin{cases}F_0, & 0\leqslant j\leqslant r-1,\\ 0, & r\leqslant j\leqslant N-1,\end{cases} \tag{2.13}$$

$$G(i)=\begin{cases}G_0, & 0\leqslant j\leqslant s-1,\\ 0, & s\leqslant j\leqslant N-1,\end{cases} \tag{2.14}$$

for some integers r and s ($1\leqslant r\leqslant N$, $1\leqslant s\leqslant N$) and some vectors F_0 and G_0 in $\mathbb{C}^n$. These special forms for F and G are assumed in what follows.

By (1.5) with $\zeta=\bar{z}$, $z=0$, and $N=r$,

$$\sum_{k=0}^{r-1} W^*(k,\bar{z})q(k+1)=\frac{\mathrm{i}}{z}\left[W^*(r,\bar{z})-I_n\right]J.$$

Thus by (1.14) and Theorem 2.1,

$$\begin{aligned}f(z)&=\sum_{k=0}^{r-1}\begin{bmatrix}0 & I_m\end{bmatrix}W^*(k,\bar{z})q(k+1)F_0\\ &=\begin{bmatrix}0 & I_m\end{bmatrix}\frac{\mathrm{i}}{z}\left[W^*(r,\bar{z})-I_n\right]JF_0\\ &=\begin{bmatrix}0 & I_m\end{bmatrix}\frac{\mathrm{i}}{z}\left\{-\mathrm{i}z\Pi^*E_r\left(I_{rm}-zA^{(r)*}\right)^{-1}\left(S^{(r)}\right)^{-1}E_r^*\Pi J\right\}JF_0\\ &=\Phi_2^*E_r\left(I_{rm}-zA^{(r)*}\right)^{-1}\left(S^{(r)}\right)^{-1}E_r^*\Pi F_0\\ &=\Phi_2^*\left(I-zA^*\right)^{-1}E_r\left(S^{(r)}\right)^{-1}E_r^*\Pi F_0.\end{aligned}$$

For the last equality we used the identity

$$E_r\left(I_{rm}-zA^{(r)*}\right)^{-1}=\left(I-zA^*\right)^{-1}E_r,$$

which is an application of (2.5) with $M=I-zA^*$ and $k=r$. Thus

$$\begin{aligned}&\langle f(z),g(z)\rangle_{L^2(\tau,N)}\\ &\quad=\int_{-\infty}^{\infty}G_0^*\Pi^*E_s\left(S^{(s)}\right)^{-1}E_s^*(I-zA)^{-1}\Phi_2\,[\mathrm{d}\tau(u)]\\ &\qquad\times\Phi_2^*(I-zA^*)^{-1}E_r\left(S^{(r)}\right)^{-1}E_r^*\Pi F_0\\ &\qquad+\sum_{k=1}^{\nu}\Big[\mathfrak{F}_k\Big(\Phi_2^*(I-zA^*)^{-1}E_r\left(S^{(r)}\right)^{-1}E_r^*\Pi F_0,\\ &\qquad\qquad\Phi_2^*(I-zA^*)^{-1}E_s\left(S^{(s)}\right)^{-1}E_s^*\Pi G_0\Big)\\ &\qquad\qquad+\widehat{\mathfrak{F}}_k\Big(\Phi_2^*(I-zA^*)^{-1}E_r\left(S^{(r)}\right)^{-1}E_r^*\Pi F_0,\\ &\qquad\qquad\Phi_2^*(I-zA^*)^{-1}E_s\left(S^{(s)}\right)^{-1}E_s^*\Pi G_0\Big)\Big]\\ &\quad=G_0^*\Pi^*E_s\left(S^{(s)}\right)^{-1}E_s^*\left\{\int_{-\infty}^{\infty}(I-Az)^{-1}\Phi_2\,[\mathrm{d}\tau(u)]\,\Phi_2^*(I-A^*z)^{-1}\right.\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\nu}\Big[\mathfrak{F}_k\big(\Phi_2^*(I-A^*z)^{-1},\Phi_2^*(I-A^*z)^{-1}\big)\\
&+\widehat{\mathfrak{F}}_k\big(\Phi_2^*(I-A^*z)^{-1},\Phi_2^*(I-A^*z)^{-1}\big)\Big]\Big\}\,E_r\big(S^{(r)}\big)^{-1}E_r^*\Pi F_0\\
=\;& G_0^*\Pi^*E_s\big(S^{(s)}\big)^{-1}E_s^*SE_r\big(S^{(r)}\big)^{-1}E_r^*\Pi F_0.
\end{aligned}
$$

Writing $t=\min(r,s)$, we find that

$$E_s\big(S^{(s)}\big)^{-1}E_s^*SE_r\big(S^{(r)}\big)^{-1}E_r^*=E_t(S^{(t)})^{-1}E_t^*.$$

For example, suppose that $s<r$; writing

$$S=\begin{bmatrix}S^{(r)} & *\\ * & *\end{bmatrix}\qquad\text{and}\qquad E_r\big(S^{(r)}\big)^{-1}E_r^*=\begin{bmatrix}\big(S^{(r)}\big)^{-1} & 0\\ 0 & 0\end{bmatrix},$$

we get

$$SE_r\big(S^{(r)}\big)^{-1}E_r^*=\begin{bmatrix}I_{rm} & 0\\ * & 0\end{bmatrix},$$

from which we obtain the assertion. Therefore

$$
\begin{aligned}
\langle f(z),g(z)\rangle_{L^2(\tau,N)}&=G_0^*\Pi^*E_t(S^{(t)})^{-1}E_t^*\Pi F_0\\
&=G_0^*\sigma(t)F_0\\
&=\sum_{k=0}^{t-1}G_0^*\left[\sigma(k+1)-\sigma(k)\right]F_0\\
&=\sum_{k=0}^{t-1}G_0^*q(k+1)F_0\\
&=\langle F,G\rangle_{\ell^2(q,N)},
\end{aligned}
$$

as was to be shown. □

With an additional nondegeneracy condition, a converse result holds.

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *be satisfied, and in addition assume that*

$$\operatorname{rank}q(k)=m,\quad k=1,\dots,N. \tag{2.15}$$

If the boundary problem (2.6) *has spectral data* $\boldsymbol{\tau}=\{\tau,\mathfrak{F}_1,\dots,\mathfrak{F}_\nu\}$, *then S is given by* (2.10).

Proof. Define S_τ to be the right-hand side of (2.10). We show that $S=S_\tau$. Notation is the same as in the proof of Theorem 2.2.

By assumption, (2.12) holds for all elements F and G of $\ell^2(q, N)$ and their transforms $f(z)$ and $g(z)$. Suppose that F and G are of the special type given by (2.11) and (2.13), (2.14). By the proof of Theorem 2.2,

$$\langle f(z), g(z)\rangle_{L^2(\tau,N)} = G_0^* \Pi^* E_s \left(S^{(s)}\right)^{-1} E_s^* S_\tau E_r \left(S^{(r)}\right)^{-1} E_r^* \Pi F_0$$

and

$$\begin{aligned}\langle F, G\rangle_{\ell^2(q,N)} &= G_0^* \Pi^* E_t (S^{(t)})^{-1} E_t^* \Pi F_0 \\ &= G_0^* \Pi^* E_s \left(S^{(s)}\right)^{-1} E_s^* S E_r \left(S^{(r)}\right)^{-1} E_r^* \Pi F_0.\end{aligned} \tag{2.16}$$

Since these expressions coincide by (2.12), the identity $S = S_\tau$ will follow if we can show that the elements of $\mathbb{C}^{mN}$ of the form

$$E_r \left(S^{(r)}\right)^{-1} E_r^* \Pi F_0, \qquad r = 1, \ldots, N, \quad F_0 \in \mathbb{C}^n, \tag{2.17}$$

are fundamental in the space. To see this, define a mapping T_0 from special elements F of $\ell^2(q, N)$ of the form (2.13) into $\mathbb{C}^{mN}$ by

$$T_0 F = E_r \left(S^{(r)}\right)^{-1} E_r^* \Pi F_0.$$

If G is given by (2.14), the identity

$$\langle S T_0 F, T_0 G\rangle_{\mathbb{C}^{mN}} = \langle F, G\rangle_{\ell^2(q,N)}$$

holds by (2.16). This identity implies that T_0 extends to a linear operator T on $\ell^2(q, N)$ into $\mathbb{C}^{mN}$ satisfying $T^* S T = I_{\ell^2(q,N)}$, and hence $\ker T = \{0\}$. Our nondegeneracy hypothesis implies that

$$\dim \ell^2(q, N) = mN,$$

and therefore T maps $\ell^2(q, N)$ onto $\mathbb{C}^{mN}$. In particular, the elements of the form (2.17) are fundamental in $\mathbb{C}^{mN}$, and the result follows. □

Corollary 2.4. *Let the assumptions of Theorem* 2.1 *be satisfied, and assume that* (2.15) *holds. The set of spectral data sets* $\boldsymbol{\tau}$ *of the boundary problem* (2.6) *coincides with the set of tuples* (1.8) *for which a space* $L^2(\boldsymbol{\tau}, N)$ *is defined and such that the identity* (2.10) *holds.*

Proof. This follows from Theorems 2.2 and 2.3. □

Corollary 2.5. *Let the assumptions of Theorem* 2.1 *be satisfied, and assume that* (2.15) *holds. Then the signature of* $\ell^2(q, N)$ *as a Pontryagin space coincides with the signature of S as a selfadjoint matrix.*

Proof. In the notation of the proof of Theorem 2.3, the identity

$$\langle STF, TG\rangle_{\mathbb{C}^{mN}} = \langle F, G\rangle_{\ell^2(q,N)}$$

holds for all $F, G \in \ell^2(q, N)$. The equality of the two signatures follows as an immediate consequence. □

We obtain a solution of the inverse spectral problem.

Theorem 2.6. *Suppose we are given data* $\boldsymbol{\tau} = \{\tau, \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}$ *such that a space* $L^2(\boldsymbol{\tau}, N)$ *is defined as in Section* 1. *Define* S *by* (2.10), *and assume that* $S^{(1)}, \ldots, S^{(N)}$ *are invertible. Define* $\Pi = \begin{bmatrix}\Phi_1 & \Phi_2\end{bmatrix}$, *where* Φ_1 *is computed from the data by*

$$\begin{aligned}\Phi_1 = -\mathrm{i}\int_{-\infty}^{\infty} A(I-uA)^{-1}\Phi_2 \, \mathrm{d}\tau(u) - \mathrm{i}\sum_{k=1}^{\nu}\big[\mathfrak{F}_k\big(I_m, \Phi_2^* A^*(I-zA^*)^{-1}\big) \\ + \widehat{\mathfrak{F}}_k\big(I_m, \Phi_2^* A^*(I-zA^*)^{-1}\big)\big],\end{aligned} \tag{2.18}$$

and Φ_2 *is as in* (2.4). *Then the boundary problem* (2.6), *where* $q(1), \ldots, q(N)$ *are constructed from* (2.7) *and* (2.8), *has spectral data* $\boldsymbol{\tau}$.

Proof. This will follow from Theorem 2.2 if we can show that S is selfadjoint and that S and Φ_1 have the forms given in (2.3) and (2.4).

Writing $\pi(z) = \Phi_2^*(I - A^*z)^{-1} = \begin{bmatrix} I_m & I_m z & I_m z^2 & \cdots & I_m z^{N-1}\end{bmatrix}$, we have

$$S = \int_{-\infty}^{\infty} \pi(u)^*[\mathrm{d}\tau(u)]\pi(u) + \sum_{k=1}^{\nu}\big[\mathfrak{F}_k(\pi(z), \pi(z)) + \widehat{\mathfrak{F}}_k(\pi(z), \pi(z))\big].$$

Here

$$\int_{-\infty}^{\infty} \pi(u)^*[\mathrm{d}\tau(u)]\pi(u) = \big[\textstyle\int_{-\infty}^{\infty} z^{i+j} \, \mathrm{d}\tau(u)\big]_{i,j=0}^{N-1}.$$

If the polynomials in the discrete part of $\boldsymbol{\tau}$ are $R_k(\lambda) = \tau_{k1}\lambda + \tau_{k2}\lambda^2 + \cdots$, $k = 1, \ldots, \nu$, then

$$\begin{aligned}&\mathfrak{F}_k(\pi(z), \pi(z)) \\ &\quad = \Big\{\pi^*(\bar{\lambda})\tau_{k1}\pi(\lambda) + \frac{\mathrm{d}}{\mathrm{d}\lambda}\big[\pi^*(\bar{\lambda})\tau_{k2}\pi(\lambda)\big] \\ &\qquad + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\big[\pi^*(\bar{\lambda})\tau_{k3}\pi(\lambda)\big] + \cdots\Big\}\Big|_{\lambda=\lambda_k} \\ &\quad = \left[\tau_{k1}\lambda_k^{i+j} + (i+j)\tau_{k2}\lambda_k^{i+j-1} + \frac{(i+j)(i+j-1)}{2!}\tau_{k3}\lambda_k^{i+j-2} + \cdots\right]_{i,j=0}^{N-1}\end{aligned}$$

and

$$\begin{aligned}&\widehat{\mathfrak{F}}_k(\pi(z), \pi(z)) \\ &\quad = \Big\{\pi^*(\bar{\lambda})\tau_{k1}^*\pi(\lambda) + \frac{\mathrm{d}}{\mathrm{d}\lambda}\big[\pi^*(\bar{\lambda})\tau_{k2}^*\pi(\lambda)\big] \\ &\qquad + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\big[\pi^*(\bar{\lambda})\tau_{k3}^*\pi(\lambda)\big] + \cdots\Big\}\Big|_{\lambda=\bar{\lambda}_k}\end{aligned}$$

$$= \left[\tau_{k1}^* \bar{\lambda}_k^{i+j} + (i+j)\tau_{k2}^* \bar{\lambda}_k^{i+j-1} + \frac{(i+j)(i+j-1)}{2!} \tau_{k3}^* \bar{\lambda}_k^{i+j-2} + \cdots \right]_{i,j=0}^{N-1}.$$

These expressions show that $S = [S_{i+j}]_{i,j=0}^{N-1}$, where $S_0, S_1, \ldots, S_{2N-2}$ are selfadjoint. In particular, S is selfadjoint.

To see that Φ_1 has the required form, notice that

$$\Phi_2^*(I - A^*z)^{-1}\Phi_2 = \Phi_2^*(I + A^*z + A^{*2}z^2 + \cdots)\Phi_2 = I_m.$$

Hence by (2.10),

$$\begin{aligned}
&-\mathrm{i}AS\Phi_2 \\
&\quad = -\mathrm{i}A \int_{-\infty}^{\infty} (I - Au)^{-1}\Phi_2\, [\mathrm{d}\tau(u)]\, \Phi_2^*(I - A^*u)^{-1} \\
&\qquad + \sum_{k=1}^{\nu} \left\{ \operatorname{Res}_{\lambda=\lambda_k} \left[A(I - A\lambda)^{-1}\Phi_2 R_k \left(\frac{1}{\lambda - \lambda_k} \right) \Phi_2^*(I - A^*\lambda)^{-1} \right] \right. \\
&\qquad\qquad \left. + \operatorname{Res}_{\lambda=\bar{\lambda}_k} \left[A(I - A\lambda)^{-1}\Phi_2 R_k^* \left(\frac{1}{\bar{\lambda} - \lambda_k} \right) \Phi_2^*(I - A^*\lambda)^{-1} \right] \right\} \Phi_2 \\
&\quad = -\mathrm{i} \int_{-\infty}^{\infty} A(I - uA)^{-1}\Phi_2\, \mathrm{d}\tau(u) \\
&\qquad -\mathrm{i} \sum_{k=1}^{\nu} \left[\mathfrak{F}_k \left(I_m, \Phi_2^* A^*(I - zA^*)^{-1} \right) + \widehat{\mathfrak{F}}_k \left(I_m, \Phi_2^* A^*(I - zA^*)^{-1} \right) \right] \\
&\quad = \Phi_1.
\end{aligned}$$

Since $S = [S_{i+j}]_{i,j=0}^{N-1}$, $\Phi_1 = -\mathrm{i}AS\Phi_2$ has the form given in (2.4).

We have shown that S is selfadjoint, and that S and Φ_1 have the forms given in (2.3) and (2.4), and so the result follows from Theorem 2.2. $\square$

Example. To show the role of the discrete parts of the boundary data, we construct a boundary problem having given spectral data

$$\boldsymbol{\tau} = \{\tau, \mathfrak{F}_1\}, \tag{2.19}$$

where $\tau(u) \equiv 0$ and $\mathfrak{F}_1$ is computed from a fixed point $\lambda_1 \neq \bar{\lambda}_1$ and a polynomial of the simplest type:

$$R_1(\lambda) = \tau_1\, \lambda, \qquad \operatorname{Re} \tau_1 \neq 0.$$

We assume the scalar case, $m = 1$, and take $N = 2$. The boundary problem is constructed by means of Theorem 2.2 with the choices

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} \tau_1 + \bar{\tau}_1 & \lambda_1\tau_1 + \bar{\lambda}_1\bar{\tau}_1 \\ \lambda_1\tau_1 + \bar{\lambda}_1\bar{\tau}_1 & \lambda_1^2\tau_1 + \bar{\lambda}_1^2\bar{\tau}_1 \end{bmatrix}$$

and

$$\Phi_1 = -\mathrm{i}\begin{bmatrix} 0 \\ \tau_1 + \bar{\tau}_1 \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We easily check that $S^{(1)}$ and $S^{(2)}$ are invertible, and

$$\begin{aligned} \left(S^{(1)}\right)^{-1} &= \frac{1}{\tau_1 + \bar{\tau}_1}, \\ \left(S^{(2)}\right)^{-1} &= \frac{1}{\Delta}\begin{bmatrix} \lambda_1^2\tau_1 + \bar{\lambda}_1^2\bar{\tau}_1 & -(\lambda_1\tau_1 + \bar{\lambda}_1\bar{\tau}_1) \\ -(\lambda_1\tau_1 + \bar{\lambda}_1\bar{\tau}_1) & \tau_1 + \bar{\tau}_1 \end{bmatrix}, \quad \Delta = (\lambda_1 - \bar{\lambda}_1)^2\tau_1\bar{\tau}_1. \end{aligned}$$

Thus (2.7) and (2.8) give

$$\begin{aligned} q(1) &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\tau_1+\bar{\tau}_1} \end{bmatrix}, \\ q(2) &= \frac{1}{(\lambda_1 - \bar{\lambda}_1)^2\tau_1\bar{\tau}_1}\begin{bmatrix} (\tau_1 + \bar{\tau}_1)^3 & -\mathrm{i}(\tau_1 + \bar{\tau}_1)(\lambda_1\tau_1 + \bar{\lambda}_1\bar{\tau}_1) \\ \mathrm{i}(\tau_1 + \bar{\tau}_1)(\lambda_1\tau_1 + \bar{\lambda}_1\bar{\tau}_1) & \lambda_1^2\tau_1 + \bar{\lambda}_1^2\bar{\tau}_1 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\tau_1+\bar{\tau}_1} \end{bmatrix}. \end{aligned}$$

By Theorem 2.1, $q(1)$ and $q(2)$ satisfy (1.2). It remains to observe that S has the form (2.10) for the given data (2.19). It therefore follows from Theorem 2.2 that the boundary problem (2.6) has spectral data (2.19). Observe that the boundary problem (2.6) is in the indefinite case, no matter how λ_1 and τ_1 are chosen ($\lambda_1 \neq \bar{\lambda}_1$, $\mathrm{Re}\,\tau_1 \neq 0$). In fact, the signs of the 22-entry in $q(1)$ and the 11-entry in $q(2)$ are opposite, and hence it cannot be true that either both $q(1)$ and $q(2)$ are nonnegative or both $q(1)$ and $q(2)$ are nonpositive.

Formulas similar to (2.10) and (2.18) for $m = 1$ were obtained under conditions different from ours in [3].

3. Jacobi systems

3.1. Connection with classical systems

The study of matrix equations (1.1) is closely related to systems for the form

$$\begin{aligned} &a(k)\psi(k, z) + b(k)\psi(k-1, z) + c(k)\psi(k-2, z) = \mathrm{i}z\psi(k-1, z), \\ &\quad k = 1, \dots, N, \end{aligned} \tag{3.1}$$

such that the associated matrix

$$L=-\mathrm{i}\begin{bmatrix} b(1) & a(1) & 0 & 0 & \cdots & 0 & 0\\ c(2) & b(2) & a(2) & 0 & \cdots & 0 & 0\\ 0 & c(3) & b(3) & a(3) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & b(N-1) & a(N-1)\\ 0 & 0 & 0 & 0 & \cdots & c(N) & b(N) \end{bmatrix} \tag{3.2}$$

satisfies a symmetry relation

$$L\mathfrak{J}=\mathfrak{J}L^*, \qquad \mathfrak{J}=\operatorname{diag}\{j_1, j_2, \dots, j_N\}, \tag{3.3}$$

for some invertible $m\times m$ matrices $j_1,\dots,j_N$ such that $j_k=j_k^*=j_k^{-1}$. In (3.1), we assume that $\psi(k,z)$ is a polynomial of degree at most k whose coefficients are $m\times m$ matrices, and $a(k),\ b(k),\ c(k)$ are $m\times m$ matrices, $k=1,\dots,N$. We take

$$\psi(-1,z)=0. \tag{3.4}$$

In this section, we outline the correspondence between systems (3.1)–(3.4) and (1.1), (1.2) and show that the spectral theories are equivalent. The results here generalize those of [4, Section 8.2].

3.2. Construction of Jacobi systems

Let us start with a system

$$\begin{cases} W(k,z)-W(k-1,z)=\mathrm{i}zJq(k)W(k-1,z), & k=1,\dots,N+1,\\ W(0,z)=I_n, \end{cases} \tag{3.5}$$

where

$$q(k)=q(k)^*, \quad q(k)Jq(k)=0, \qquad k=1,\dots,N+1. \tag{3.6}$$

We show how to construct a system of the type (3.1)–(3.4).

Proposition 3.1. *The general form of matrices* $q(1),\dots,q(N+1)$ *satisfying* (3.6) *is*

$$q(k)=\begin{bmatrix} p_1(k)\\ p_2(k)\end{bmatrix} j_k \begin{bmatrix} p_1^*(k) & p_2^*(k)\end{bmatrix}, \tag{3.7}$$

where j_k *is an invertible* $m\times m$ *matrix such that* $j_k=j_k^*=j_k^{-1}$ *and* $p_1(k)$ *and* $p_2(k)$ *are* $m\times m$ *matrices such that*

$$p_1^*(k)p_2(k)+p_2^*(k)p_1(k)=0, \quad k=1,\dots,N+1. \tag{3.8}$$

For each fixed k, the condition $\operatorname{rank} q(k)=m$ *holds if and only if* $\operatorname{rank}[p_1^*(k) \quad p_2^*(k)] = m$.

Proof. In fact, it is not hard to see that the general form of an $n\times n$ matrix q satisfying $q^*=q$ and $qJq=0$ is

$$q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} j \begin{bmatrix} p_1^* & p_2^* \end{bmatrix},$$

where for some $r \leqslant m$, j is an invertible $r \times r$ matrix satisfying $j = j^* = j^{-1}$ and p_1 and p_2 are $m \times r$ matrices such that $p_1^* p_2 + p_2^* p_1 = 0$ and the only $x \in \mathbb{C}^m$ such that $p_1 x = p_2 x = 0$ is $x = 0$. In particular, rank $q = r \leqslant m$. The assertions are readily deduced from this representation applied to $q = q(k)$ for any fixed k. □

Theorem 3.2. *Let a system* (3.5), (3.6) *be given. Represent* $q(1), \ldots, q(N+1)$ *in the form* (3.7), (3.8), *and choose* $q(0)$ *of the same form with* $p_1(0) = I_m$ *and* $p_2(0) = 0$. *Assume that*

$$\det\,[p_1^*(k+1)p_2(k) + p_2^*(k+1)p_1(k)] \neq 0, \quad k = 0, \ldots, N. \tag{3.9}$$

Write

$$\begin{aligned} A(k) &= j_{k+1}\left[\, p_1^*(k+1)p_2(k) + p_2^*(k+1)p_1(k)\right], \\ B(k) &= j_k\left[\, p_1^*(k)p_2(k+1) + p_2^*(k)p_1(k+1)\right], \end{aligned}$$

$k = 0, \ldots, N$. *Then the functions* $\psi(-1, z) = 0$ *and*

$$\psi(k, z) = j_{k+1}\left[p_1^*(k+1) \quad p_2^*(k+1)\right] W(k, z) \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad k = 0, \ldots, N,$$

satisfy (3.1)–(3.4), *where for all* $k = 1, \ldots, N$,

$$\begin{aligned} a(k) &= A^{-1}(k), \\ b(k) &= -A^{-1}(k)\, j_{k+1}\left[p_1^*(k+1)p_2(k-1) + p_2^*(k+1)p_1(k-1)\right] A^{-1}(k-1), \\ c(k) &= -B^{-1}(k-1). \end{aligned}$$

The hypothesis (3.9) implies that we are in the full-rank case, that is, all of the matrices $q(1), \ldots, q(N+1)$ have rank m.

Proof of Theorem 3.2 (*beginning*). We obtain all of the conclusions of Theorem 3.2 except for one step in the proof of (3.3), and this will be given later.

For later calculations, observe that the relation (3.8) holds for $k = 0$ as well as $k = 1, \ldots, N+1$. Write

$$W(k, z) = \begin{bmatrix} W_{11}(k, z) & W_{12}(k, z) \\ W_{21}(k, z) & W_{22}(k, z) \end{bmatrix}, \quad k = 0, \ldots, N+1.$$

By the definiton of $\psi(k, z)$,

$$\begin{aligned} \psi(k, z) &= j_{k+1}\left[\, p_1^*(k+1)W_{12}(k, z) + p_2^*(k+1)W_{22}(k, z)\,\right], \\ &\quad k = 0, \ldots, N. \end{aligned} \tag{3.10}$$

By (3.5), for any $k = 1, \ldots, N+1$,

$$\begin{bmatrix} * & W_{12}(k,z) - W_{12}(k-1,z) \\ * & W_{22}(k,z) - W_{22}(k-1,z) \end{bmatrix}$$
$$= \mathrm{i}z \begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix} j_k \begin{bmatrix} p_1^*(k) & p_2^*(k) \end{bmatrix} \begin{bmatrix} * & W_{12}(k-1,z) \\ * & W_{22}(k-1,z) \end{bmatrix}$$
$$= \mathrm{i}z \begin{bmatrix} * & p_2(k)\psi(k-1,z) \\ * & p_1(k)\psi(k-1,z) \end{bmatrix},$$

that is,

$$\begin{cases} W_{12}(k,z) = W_{12}(k-1,z) + \mathrm{i}z\, p_2(k)\psi(k-1,z), \\ W_{22}(k,z) = W_{22}(k-1,z) + \mathrm{i}z\, p_1(k)\psi(k-1,z). \end{cases} \tag{3.11}$$

Substituting these expressions into (3.10), we obtain

$$\psi(k,z) - j_{k+1}\left[p_1^*(k+1)W_{12}(k-1,z) + p_2^*(k+1)W_{22}(k-1,z)\right]$$
$$= \mathrm{i}z\, A(k)\psi(k-1,z), \quad k = 1, \dots, N. \tag{3.12}$$

We show next that for all $k = 2, \dots, N+1$,

$$\begin{cases} j_k p_1^*(k)W_{12}(k-1,z) + j_k p_2^*(k)W_{22}(k-1,z) = \psi(k-1,z), \\ j_{k-1}p_1^*(k-1)W_{12}(k-1,z) + j_{k-1}p_2^*(k-1)W_{22}(k-1,z) \\ \quad = \psi(k-2,z). \end{cases} \tag{3.13}$$

In fact, the first of these equations follows from (3.10) with k replaced by $k-1$. To prove the second, use (3.11) with k replaced by $k-1$ and (3.8) to obtain

$$j_{k-1}p_1^*(k-1)W_{12}(k-1,z) + j_{k-1}p_2^*(k-1)W_{22}(k-1,z)$$
$$= j_{k-1}p_1^*(k-1)\left[W_{12}(k-2,z) + \mathrm{i}z\, p_2(k-1)\psi(k-2,z)\right]$$
$$+ j_{k-1}p_2^*(k-1)\left[W_{22}(k-2,z) + \mathrm{i}z\, p_1(k-1)\psi(k-2,z)\right]$$
$$= j_{k-1}\left[p_1^*(k-1)W_{12}(k-2,z) + p_2^*(k-1)W_{22}(k-2,z)\right]$$
$$= \psi(k-2,z).$$

The relations (3.13) can be solved for $W_{12}(k-1,z)$ and $W_{22}(k-1,z)$ by noting that

$$\begin{bmatrix} j_k p_1^*(k) & j_k p_2^*(k) \\ j_{k-1}p_1^*(k-1) & j_{k-1}p_2^*(k-1) \end{bmatrix} \begin{bmatrix} p_2(k-1) & p_2(k) \\ p_1(k-1) & p_1(k) \end{bmatrix}$$
$$= \begin{bmatrix} A(k-1) & 0 \\ 0 & B(k-1) \end{bmatrix},$$

by the definitions of $A(k-1)$ and $B(k-1)$ and (3.8), and hence

$$\begin{bmatrix} j_k p_1^*(k) & j_k p_2^*(k) \\ j_{k-1}p_1^*(k-1) & j_{k-1}p_2^*(k-1) \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} p_2(k-1) & p_2(k) \\ p_1(k-1) & p_1(k) \end{bmatrix} \begin{bmatrix} A^{-1}(k-1) & 0 \\ 0 & B^{-1}(k-1) \end{bmatrix}.$$

The inverses required here exist by our hypothesis (3.9). In this way (3.13) yields

$$\begin{aligned} W_{12}(k-1,z) &= p_2(k-1)A^{-1}(k-1)\psi(k-1,z) \\ &\quad + p_2(k)B^{-1}(k-1)\psi(k-2,z), \\ W_{22}(k-1,z) &= p_1(k-1)A^{-1}(k-1)\psi(k-1,z) \\ &\quad + p_1(k)B^{-1}(k-1)\psi(k-2,z), \end{aligned}$$

$k = 2, \ldots, N+1$. On substituting these expressions into (3.12) and simplifying, we obtain (3.1), at least for $k = 2, \ldots, N$. The case $k = 1$ in (3.1) may be checked by explicitly calculating all of the quantities.

It remains to verify the identity (3.3). It is immediate from our formulas that $c(k) = -j_k a^*(k-1) j_{k-1}$, $k = 2, \ldots, N$. To complete the proof, we must show that $b(k) = -j_k b^*(k) j_k$, $k = 1, \ldots, N$. This final step is completed below. □

Both the final step in the proof of Theorem 3.2 and the converse result require algebraic constructions which are adapted from the theory of Toda chains ([4, Chapter 10] and [6]). In the definite case, the required results are given in [4, pp. 116–118]. We modify these formulas for the indefinite case.

We assume the same hypotheses as in Theorem 3.2. A key role is played by invertible matrices $\beta(1), \ldots, \beta(N+1)$ which are defined by

$$\begin{cases} \beta(1) = p_1^*(1)p_1(1) + p_2^*(1)p_2(1), \\ \beta(k) = \left[p_1^*(k)p_2(k-1) + p_2^*(k)p_1(k-1)\right] j_{k-1}\beta(k-1), \\ \quad k = 2, \ldots, N+1. \end{cases} \tag{3.14}$$

Set

$$U(0) = \begin{bmatrix} p_1(1) & p_2(1)\beta^{-1}(1) \\ p_2(1) & p_1(1)\beta^{-1}(1) \end{bmatrix}$$

and

$$U(k) = \begin{bmatrix} p_2(k)j_k\beta(k) & p_2(k+1)\beta^{*-1}(k+1) \\ p_1(k)j_k\beta(k) & p_1(k+1)\beta^{*-1}(k+1) \end{bmatrix}, \quad k = 1, \ldots, N. \tag{3.15}$$

Straightforward calculations show that

$$\begin{aligned} &U(k)\mathscr{E} = Jq(k)U(k-1), \quad k = 1, \ldots, N, \\ &U^*(k)JU(k) = J, \quad k = 0, \ldots, N, \end{aligned}$$

where

$$\mathscr{E} = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}.$$

By virtue of these identities, we obtain a new system

$$\widetilde{W}(k,z) = U^{-1}(k)W(k,z)U(0), \quad k = 0,\ldots,N, \tag{3.16}$$

satisfying

$$\begin{cases} \widetilde{W}(k,z) = [\mathrm{i}z\mathscr{E} + \xi(k)]\,\widetilde{W}(k-1,z), & k = 1,\ldots,N, \\ \widetilde{W}(0,z) = I_n, \end{cases} \tag{3.17}$$

where

$$\xi(k) = U^{-1}(k)U(k-1), \quad k = 1,\ldots,N. \tag{3.18}$$

We find that for all $k = 1,\ldots,N$,

$$\xi(k) = \begin{bmatrix} R(k) & C(k) \\ C^{-1}(k) & 0 \end{bmatrix},$$

$$R(k) = \begin{cases} \beta^{-1}(2)\left[p_2^*(2)p_2(1) + p_1^*(2)p_1(1)\right], & k = 1, \\ \beta^{-1}(k+1)[p_2^*(k+1)p_1(k-1) \\ \quad +p_1^*(k+1)p_2(k-1)]j_{k-1}\beta(k-1), & k = 2,\ldots,N, \end{cases}$$

$$C(k) = \beta^{-1}(k)j_k\beta^{*-1}(k).$$

Since $U(k)$ is J-unitary for each k, so is $\xi(k)$, and this implies the identity

$$C(k)R^*(k) + R(k)C(k) = 0, \quad k = 1,\ldots,N, \tag{3.19}$$

which, in particular, allows us to complete the proof of Theorem 3.2.

Proof of Theorem 3.2 (*completion*). For $k = 2,\ldots,N$, we have

$$b(k) = -j_k\beta(k)R(k)\beta^{-1}(k)j_k.$$

The identity $b(k) = -j_k b^*(k)j_k$ is thus seen to hold by (3.19). We verify $b(1) = -j_1 b^*(1)j_1$ by direct calculation from the definitions. □

To facilitate the converse direction, we note some additional formulas that hold in the environment of Theorem 3.2. Define matrices

$$T(k) = \begin{bmatrix} p_2(k) & p_2(k-1) \\ p_1(k) & p_1(k-1) \end{bmatrix}, \quad k = 2,\ldots,N. \tag{3.20}$$

For such k,

$$\begin{aligned} T^*(k)JT(k) &= \begin{bmatrix} p_2^*(k) & p_1^*(k) \\ p_2^*(k-1) & p_1^*(k-1) \end{bmatrix} \begin{bmatrix} p_1(k) & p_1(k-1) \\ p_2(k) & p_2(k-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \beta(k)\beta^{-1}(k-1)j_{k-1} \\ j_{k-1}\beta^{*-1}(k-1)\beta^*(k) & 0 \end{bmatrix}. \end{aligned} \tag{3.21}$$

In particular, the matrices $T(2),\ldots,T(N)$ are invertible. For the same values of k,

$$
\begin{aligned}
&\left[p_1^*(k+1) \quad p_2^*(k+1)\right] T(k)\\
&\quad = \left[p_1^*(k+1)p_2(k) + p_2^*(k+1)p_1(k) \right.\\
&\qquad\qquad \left. p_1^*(k+1)p_2(k-1) + p_2^*(k+1)p_1(k-1)\right]\\
&\quad = \left[\beta(k+1)\beta^{-1}(k)j_k \quad \beta(k+1)R(k)\beta^{-1}(k-1)j_{k-1}\right]\\
&\quad = j_{k+1}\left[a^{-1}(k) \quad -a^{-1}(k)b(k)a^{-1}(k-1)\right],
\end{aligned} \tag{3.22}
$$

and hence we obtain the recursive formula

$$
\begin{aligned}
&\left[p_1^*(k+1) \quad p_2^*(k+1)\right] = j_{k+1}\left[a^{-1}(k) \;\; -a^{-1}(k)b(k)a^{-1}(k-1)\right] T^{-1}(k),\\
&\quad k = 2, \ldots, N.
\end{aligned} \tag{3.23}
$$

It is not hard to extend the recursive formula to $k = 1$ as well, but this will not be needed in what follows.

3.3. Converse direction

The Jacobi system (3.1)–(3.4) constructed in Theorem 3.2 has additional properties: $a(1), \ldots, a(N)$ are invertible, and $\psi(0, z) \equiv \psi_0$ is a constant invertible matrix. If we add these conditions, then the construction in Theorem 3.2 is reversible.

Theorem 3.3. *Let* (3.1)–(3.4) *be a Jacobi system such that the matrices* $a(1), \ldots,$ $a(N)$ *are invertible and* $\psi(0, z) \equiv \psi_0$ *is a constant invertible matrix. Then the system* (3.1)–(3.4) *arises from the construction in Theorem* 3.2 *from some canonical system* (3.5), (3.6). *Explicitly, this means that it is possible to find matrices* $q(1), \ldots, q(N+1)$ *of the form* (3.7) *satisfying* (3.8) *and* (3.9), *where* $p_1(0) = I_m$ *and* $p_2(0) = 0$, *such that*

$$
\psi(k, z) = j_{k+1}\left[p_1^*(k+1) \quad p_2^*(k+1)\right] W(k, z)\begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad k = 0, \ldots, N,
$$

where $W(k, z)$, $k = 1, \ldots, N+1$, *is the solution of* (3.5), *and for all* $k = 1, \ldots, N$,

$$
\begin{aligned}
&a(k) = A^{-1}(k),\\
&b(k) = -A^{-1}(k)j_{k+1}\left[p_1^*(k+1)p_2(k-1) + p_2^*(k+1)p_1(k-1)\right]A^{-1}(k-1),\\
&c(k) = -B^{-1}(k-1),
\end{aligned}
$$

where

$$
\begin{aligned}
A(k) &= j_{k+1}\left[p_1^*(k+1)p_2(k) + p_2^*(k+1)p_1(k)\right],\\
B(k) &= j_k\left[p_1^*(k)p_2(k+1) + p_2^*(k)p_1(k+1)\right],
\end{aligned} \quad k = 0, \ldots, N.
$$

Proof. To prove the theorem, we shall inductively construct $q(1), \ldots, q(N+1)$ satisfying the hypotheses of Theorem 3.2 such that the resulting Jacobi system is

(3.1)–(3.4). For the purpose of the proof, we denote the quantities generated by the construction in Theorem 3.2 up to some stage r by

$$\tilde{a}(k), \tilde{b}(k), \tilde{c}(k), \quad k = 1, \ldots, r,$$
$$\tilde{\psi}(k, z), \quad k = -1, \ldots, r.$$

We shall define $q(1), \ldots, q(r+1)$ such that the hypotheses of Theorem 3.2 are satisfied up to this stage, and

$$\tilde{a}(k) = a(k), \quad \tilde{b}(k) = b(k), \quad \tilde{c}(k) = c(k), \quad k = 1, \ldots, r,$$
$$\tilde{\psi}(k, z) = \psi(k, z), \quad k = -1, \ldots, r.$$

By (3.7), we can specify $q(1), \ldots, q(r+1)$ by choosing $p_1(k)$, $p_2(k)$, $k = 0, \ldots, r+1$. The inductive construction is carried out for $r = 1, \ldots, N$.

We show that for $r = 1$, all of the requirements are met with the choices

$$p_1(0) = I_m,$$
$$p_2(0) = 0,$$
$$p_1^*(1) = 0,$$
$$p_2^*(1) = j_1\psi_0,$$
$$p_1^*(2) = j_2 a^{-1}(1) p_2^{-1}(1),$$
$$p_2^*(2) = -j_2 a^{-1}(1) b(1) j_1 p_2^*(1).$$

Without difficulty we check the hypotheses of Theorem 3.2 up to this stage, that is, (3.8) holds for $k = 1, 2$, and (3.9) holds for $k = 0, 1$; here the case $k = 2$ in (3.8) follows from the symmetry of $b(1)$: $b^*(1) = -j_1 b(1) j_1$. The assigned values of $p_1(1)$, $p_2(1)$ assure that $\tilde{\psi}(0, z) = \psi(0, z)$. The definitions of $p_1(2)$, $p_2(2)$ guarantee that $\tilde{a}(1) = a(1)$ and $\tilde{b}(1) = b(1)$. Since $\tilde{\psi}(-1, z) = \psi(-1, z) = 0$, $\tilde{c}(1)$ and $c(1)$ play no role, and we are free to redefine $c(1)$ so that $\tilde{c}(1) = c(1)$. Since

$$a(1)\psi(1, z) + b(1)\psi(0, z) = \mathrm{i}z\psi(0, z),$$
$$\tilde{a}(1)\tilde{\psi}(1, z) + \tilde{b}(1)\tilde{\psi}(0, z) = \mathrm{i}z\tilde{\psi}(0, z),$$

where $\tilde{a}(1) = a(1)$ is an invertible operator, $\tilde{\psi}(1, z) = \psi(1, z)$.

Suppose that $r \geqslant 2$ and that the construction has proceeded up to the stage $r-1$, that is, $p_1(k)$, $p_2(k)$ have been chosen for $k = 1, \ldots, r$ such that

$$p_1^*(k) p_2(k) + p_2^*(k) p_1(k) = 0, \quad k = 1, \ldots, r, \tag{3.24}$$
$$\det[p_1^*(k+1) p_2(k) + p_2^*(k+1) p_1(k)] \neq 0, \quad k = 0, \ldots, r-1, \tag{3.25}$$
$$\tilde{a}(k) = a(k), \quad \tilde{b}(k) = b(k), \quad \tilde{c}(k) = c(k), \quad k = 1, \ldots, r-1, \tag{3.26}$$
$$\tilde{\psi}(k, z) = \psi(k, z), \quad k = -1, \ldots, r-1. \tag{3.27}$$

Define $p_1(r+1)$, $p_2(r+1)$ by (3.23) with $k=r$, where $T(r)$ is defined by (3.20) with $k=r$. To see that the definition is meaningful, we must show that $T(r)$ is invertible. In fact, our inductive hypothesis (3.25) allows us to define invertible operators $\beta(1),\dots,\beta(r)$ by (3.14), and then $T(r)$ is invertible by (3.21) with $k=r$. At the final stage the definitions of $p_1(N+1)$, $p_2(N+1)$ by means of (3.20) with $k=N$ depend on an operator j_{N+1}; this signature operator is not given in the system (3.1)–(3.4), but any choice will work because the operator is not needed for the conclusions.

Proof of (3.25) *for* $k=r$. By the definition of $p_1(r+1)$, $p_2(r+1)$,

$$p_1^*(r+1)p_2(r)+p_2^*(r+1)p_1(r)=j_{r+1}a^{-1}(r)$$

is an invertible matrix, that is, (3.25) holds for $k=r$.

Proof of (3.24) *for* $k=r+1$. This will be deduced from the symmetry relation

$$j_r b^*(r)+b(r)j_r=0. \tag{3.28}$$

Since (3.25) holds for $k=r$, we may define an invertible operator $\beta(r+1)$ as in (3.14). Define

$$C(r)=\beta^{-1}(r)j_r\beta^{*-1}(r), \tag{3.29}$$

$$\begin{aligned}R(r)=\beta^{-1}(r+1)&\left[p_2^*(r+1)p_1(r-1)+p_1^*(r+1)p_2(r-1)\right]\\&\times j_{r-1}\beta(r-1).\end{aligned} \tag{3.30}$$

We find that

$$a^{-1}(r)=\beta(r+1)\beta^{-1}(r)j_r, \tag{3.31}$$

$$-a^{-1}(r)b(r)a^{-1}(r-1)=\beta(r+1)R(r)\beta^{-1}(r-1)j_{r-1} \tag{3.32}$$

and so the definition of $p_1(r+1)$ and $p_2(r+1)$ can be written

$$\begin{aligned}&\left[p_1^*(r+1)\quad p_2^*(r+1)\right]\\&\quad=j_{k+1}\left[\beta(r+1)\beta^{-1}(r)j_r\quad \beta(r+1)R(r)\beta^{-1}(r-1)j_{r-1}\right]T^{-1}(r).\end{aligned}$$

We caution that although the definitions of $C(r)$ and $R(r)$ have the same form as in the previous section, we cannot assert on the basis of the earlier discussion that

$$C(r)R^*(r)+R(r)C(r)=0, \tag{3.33}$$

because this identity depends on the *J*-unitarity of the matrix $U(r)$ defined by (3.15) with $k=r$, and the proof of unitarity requires what we are trying to show here. Nevertheless, (3.33) is true and is now a consequence of (3.28). On expanding

$$\begin{aligned}&p_1^*(r+1)p_2(r+1)+p_2^*(r+1)p_1(r+1)\\&\quad=\left[p_1^*(r+1)\quad p_2^*(r+1)\right]J\begin{bmatrix}p_1(r+1)\\p_2(r+1)\end{bmatrix}\end{aligned}$$

and simplifying using (3.21) with $k = r$ and (3.33), we obtain the desired conclusion that (3.24) holds for $k = r + 1$.

Proof of (3.26) *for* $k = r$. The identities $\tilde{a}(r) = a(r)$ and $\tilde{b}(r) = b(r)$ follow quickly from the definitions. To show that $\tilde{c}(r) = c(r)$, observe that $\tilde{c}(r) = -j_r\tilde{a}^*(r-1)j_{r-1} = -j_r a^*(r-1)j_{r-1} = c(r)$ by (3.26) and the symmetry relation $c(r) = -j_r a^*(r-1)j_{r-1}$.

Proof of (3.27) *for* $k = r$. Since

$$a(r)\psi(r,z) + b(r)\psi(r-1,z) + c(r)\psi(r-2,z) = \mathrm{i}z\psi(r-1,z),$$
$$\tilde{a}(r)\tilde{\psi}(r,z) + \tilde{b}(r)\tilde{\psi}(r-1,z) + \tilde{c}(r)\tilde{\psi}(r-2,z) = \mathrm{i}z\tilde{\psi}(r-1,z),$$

where corresponding coefficients are equal and $\tilde{a}(r) = a(r)$ is invertible, $\tilde{\psi}(r,z) = \psi(r,z)$.

This completes the inductive step, and the result follows. □

3.4. *Spectral theory*

Consider a Jacobi system (3.1) satisfying (3.3). Define a transform $\widetilde{V}_N f = \tilde{f}(z)$ by

$$\tilde{f}(z) = \sum_{k=0}^{N-1} \psi^*(k,\bar{z}) f(k)$$

for all

$$f = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}, \qquad f(0), \ldots, f(N-1) \in \mathbb{C}^m.$$

By *spectral data* for the system we mean a tuple $\boldsymbol{\tau} = \{\tau, \mathfrak{F}_1, \ldots, \mathfrak{F}_\nu\}$ as in (1.8) such that $\widetilde{V}_N$ is an isometry relative to the $\mathfrak{J}$-inner product, that is, the identity

$$\sum_{k=0}^{N-1} f^*(k) j_k f(k) = \left\langle \tilde{f}(z), \tilde{f}(z) \right\rangle_{L^2(\boldsymbol{\tau})} \tag{3.34}$$

holds for all $f(0), \ldots, f(N-1) \in \mathbb{C}^m$.

The spectral theory for the Jacobi system is derivable from that of the corresponding system (3.5) with $\begin{bmatrix} D_1 & D_2 \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix}$. For suppose (3.5) has spectral data $\boldsymbol{\tau}$. Given *f* as above for some $f(0), \ldots, f(N-1) \in \mathbb{C}^m$, we can write

$$f(k) = \begin{bmatrix} p_1^*(k+1) & p_2^*(k+1) \end{bmatrix} G(k),$$

where

$$G = \begin{bmatrix} G(0) \\ \vdots \\ G(N-1) \end{bmatrix}, \qquad G(0), \ldots, G(N-1) \in \mathbb{C}^n.$$

By (3.10), $\tilde{f}(z) = \widetilde{V}_N f$ is given by

$$\begin{aligned}(\widetilde{V}_N f)(z) &= \sum_{k=0}^{N-1} \left[W_{12}^*(k,\bar{z})p_1(k+1) + W_{22}^*(k,\bar{z})p_2(k+1)\right] \\ &\qquad \times j_{k+1}\left[p_1^*(k+1) \quad p_2^*(k+1)\right] G(k) \\ &= \sum_{k=0}^{N-1} \left[0 \quad I_m\right] W^*(k,\bar{z})q(k+1)G(k). \\ &= (V_N G)(z).\end{aligned}$$

Therefore

$$\begin{aligned}\langle \tilde{f}(z), \tilde{f}(z)\rangle_{L^2(\boldsymbol{\tau})} &= \langle V_N G, V_N G\rangle_{L^2(\boldsymbol{\tau})} = \langle G, G\rangle_{\ell^2(q,N)} \\ &= \sum_{k=0}^{N-1} G(k)^* q(k+1)G(k) \\ &= \sum_{k=0}^{N-1} G(k)^* \begin{bmatrix} p_1(k+1) \\ p_2(k+1)\end{bmatrix} j_{k+1}\left[p_1^*(k+1) \quad p_2^*(k+1)\right] G(k) \\ &= \sum_{k=0}^{N-1} f^*(k) j_k f(k).\end{aligned}$$

Thus the original problem also has the spectral data $\boldsymbol{\tau}$. These steps are reversible.

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